HOMOGENEOUS EXTENSIONS OF POSITIVE LINEAR OPERATORS

BY

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1. **Introduction.** A positive linear operator is (roughly speaking) a countably additive, order preserving, σ-finite linear mapping ϕ from one function space, F, to another, F'.(1) (For precise definitions, see §2 below. We assume in particular that F and F' satisfy the countable chain condition.) It has been shown in [4] that a normal form representation can be given for ϕ: if the function space F' consists of the "measurable" functions (modulo "null" functions) on a space X, then F is isomorphic to a subspace of the space F^* of "measurable" functions on $X \times Y$, where Y is an ordinary numerical measure space, and ϕ can be extended to a positive linear operator ϕ^* from F^* to F', in such a way that (to within the isomorphism mentioned) $\phi^* f = g$, where $g(x) = \int_Y f(x, y)dy$.

We are concerned here with the case in which $F' = F$. It is now of importance (for instance in ergodic theory) to consider the iterates of ϕ; and the normal form representation just mentioned has now the drawback that the isomorphism imbedding F in F^* interferes with the description of these iterates. The present paper takes a first step towards obtaining a more satisfactory representation of ϕ and its iterates.

Given a positive linear operator ϕ from F to F, we shall show (Theorem 1, 4.1) that the function space F can be imbedded in a larger space F^*, and the operator ϕ extended to a positive linear operator ϕ^* from F^* to F^*, in such a way that the extended operator has the following property, which we call "full homogeneity": For each characteristic function $\chi \in F^*$, and each function $g \in F^*$ such that $0 \leqq g \leqq \phi^* \chi$, there exists a characteristic function χ' in F^* such that $\chi' \leqq \chi$ and $\phi^* \chi' = g$.(2) It follows that the iterates of the extended operator ϕ^* will also be fully homogeneous, and therefore σ-finite. (Even in simple cases, the iterates of ϕ itself may fail to be σ-finite(3).) A routine application of the results of [4] would then lead easily to a simultaneous representation theorem for ϕ and its iterates; however, a sharper theorem can be

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(1) Or from one conditionally complete vector σ-lattice, satisfying the countable chain condition, to another; cf. [4, p. 156].

(2) To improve the legibility of formulae, we often omit brackets, writing (as here) $\phi^*\chi$ for $\phi^*(\chi)$, and later Rx for $R(x)$, etc.

(3) For example, let F be the space of measurable functions $f(x, y)$ modulo null functions on the plane (with ordinary measure), and let $\phi f = g$ where $g(x, y) = \int_{-\infty}^{\infty} f(x, t)dt$ (independent of y). Then ϕ is an F-integral on F (and in particular is σ-finite), but if $f \in F^+$ and $\phi^2 f$ is finite, then $f = 0$, whence ϕ^2 is not σ-finite.

deduced with more trouble, so we leave this application for a later paper. Meanwhile we show (Theorem 3, §7) that when F arises from a genuine numerical measure (that is, F is the space of measurable functions modulo null functions on a measure space) then the extended space F^* in Theorem 1 can also be taken to arise from a numerical measure. The deduction of Theorem 3 depends on a property of measure algebras (Theorem 2, §6) which may be of independent interest: Given a σ-subalgebra A of a measure algebra (E, μ), and given a σ-finite measure ν on A which is equivalent to μ on A (that is, ν vanishes only for the zero element of A), there exists a σ-finite measure ν^* on E which extends ν on A and is equivalent to μ on E.

The technique employed for the proofs makes considerable use of representation spaces and of continuous functions on them; thus, after giving the notation and some preliminary results in §2, we collect some results on the representation spaces of an algebra and a complete subalgebra of it in §3. Theorem 1 and its proof occupy §§4 and 5, Theorem 2 is dealt with in §6, and Theorem 3 in §7. The background material and a few specific results are quoted without proof from [2–5]; apart from this the present paper is largely self-contained.

2. Notation and preliminaries.

2.1 *Algebras and representation spaces.* In general we use the same notations as in [2; 4], an acquaintance with which is assumed. The term "algebra" always means "Boolean algebra"; if E is an algebra, the symbols o and e denote the zero and unit elements of E respectively, and $-x$ denotes the complement of $x \in E$; and the symmetric difference of x, $y \in E$ (written $x +_2 y$ in [2; 4]) is here written as $x+y$.

The *representation space* of an algebra E is the space R of ultrafilters on E; to each $x \in E$ corresponds the set $Rx \subset R$ consisting of those ultrafilters which contain x (thus $Ro = \phi$, $Re = R$), and T is topologised by taking the sets Rx as a basis; R is compact Hausdorff, and each Rx is both open and closed. The correspondence $x \leftrightarrow Rx$ is a finite isomorphism between E and the algebra of all open-closed subsets of R. We write $\mathcal{B}^*R$ for the family of Borel subsets of R, $\mathcal{B}R$ for the family of "restricted Borel sets" (the Borel field generated by the open-closed sets of R), and $\mathcal{K}R$ for the family of first category subsets of R. We have:

(1) If $X \in \mathcal{B}^*R$, $X = G + H$ where G is open and $H \in \mathcal{K}R$.

(Here again $+$ denotes symmetric difference.) As the method of proof of (1) ("Borel induction") will be used frequently in what follows, we sketch it: The family of all sets of the form $G+H$, where G is open and $H \in \mathcal{K}R$, is closed under complementation and under countable unions; hence it is a Borel field containing all open sets, and so it contains $\mathcal{B}^*R$.

Similarly, if E is a σ-algebra, we have

(2) If $X \in \mathcal{B}R$, $X = Rx + H$ where $x \in E$ and $H \in \mathcal{K}R$.

It follows that the σ-algebra E is isomorphic to $\mathcal{B}R/\mathcal{K}R$; if further E is a complete algebra (as it will be if it satisfies the countable chain condition) then this σ-isomorphism is necessarily complete. If E is complete, (2) applies to all sets in $\mathfrak{B}^*R$, so that E is (completely) isomorphic to $\mathfrak{B}^*R/\mathfrak{K}R$.

2.2 *Subalgebras.* Let A be a subalgebra of an arbitrary algebra E, and let S, R be their respective representation spaces. There is a natural mapping ξ of R onto S, obtained as follows: each point of R is an ultrafilter $\mathfrak{U}$ on E, and its trace $\mathfrak{U}\cap A$ on A is an ultrafilter on A; we take $\xi(\mathfrak{U})=\mathfrak{U}\cap A$. It is easily verified that

$$(1)\qquad \xi^{-1}(Sy) = Ry, \qquad (y \in A),$$

so that ξ is continuous. If further B is a subalgebra of A, with T as its representation space, and ξ', ξ'' are the corresponding mappings from R to T, S to T, then clearly

$$(2)\qquad \xi' = \xi''\xi.$$

Let $\mathfrak{B}_o$ be the Borel field (of subsets of R) generated by the sets Ry, $y\in B$; a "Borel induction" argument shows that

$$(3)\qquad \mathfrak{B}R \supset \mathfrak{B}_o = \xi^{-1}\mathfrak{B}S.$$

2.3 *Complete subalgebras.* A subalgebra A of an arbitrary algebra E will be called a *complete subalgebra of* E if, for every $H \subset A$, the supremum $\bigvee H=\bigvee\{h \mid h\in H\}$ *of* H *in* E exists and belongs to A; thus A (but not necessarily E) is then itself a complete algebra. An isomorphism θ of an algebra B onto a subalgebra A of E is "complete with respect to E" if A is a complete subalgebra of E; this implies that B is itself a complete algebra and that, for every $H\subset B$, $\theta(\bigvee H)=\bigvee(\theta H)$ (the supremum in A or E).

Now suppose A is a complete subalgebra of an arbitrary algebra E, and let S, R, be their respective representation spaces, and ξ the mapping of R onto S introduced in 2.2. For each $x\in E$, write $x^*=\bigwedge\{y \mid y\in A,\ y\geqq x\}$ (the infimum referring to E, but $x^*\in A$). It is now easily verified that

$$(1)\qquad \xi Rx = Sx^*,$$

so that ξ is now *open* as well as continuous (and so closed, as R, S are compact Hausdorff). It follows that

$$(2)\qquad \xi^{-1}\mathfrak{K}S \subset \mathfrak{K}R.$$

We deduce:

(3) If $H\in\mathfrak{B}^*S$ and $\xi^{-1}H\in\mathfrak{K}R$, then $H\in\mathfrak{K}S$.

For, by 2.1(2), $H=G+Z$ where G is open and $Z\in\mathfrak{K}S$. Hence $\xi^{-1}H=\xi^{-1}G+\xi^{-1}Z$, showing that the open set $\xi^{-1}G$ is (from (2)) of first category in the compact space R; hence $\xi^{-1}G=\emptyset$, giving $H=Z$ as required.

These results, together with 2.2(3), give

$$(4)\qquad \mathfrak{K}R \cap \mathfrak{B}_o = \xi^{-1}(\mathfrak{K}S \cap \mathfrak{B}S).$$

2.4 *Function spaces.* Let S be any set, $\mathcal{B}$ any Borel field of subsets of S, and $\mathcal{N}$ any σ-ideal of subsets of S. By a "function" f on S we mean an extended-real function (so that, for each $s\in S$, $f(s)$ is real or ∞ or $-\infty$). We make the convention that $0\cdot\infty=0$, but $\infty-\infty$ is not defined. $\mathcal{F}(\mathcal{B})$ denotes the collection of "$\mathcal{B}$-measurable" functions on S, that is, of functions f such that, for each real ρ, $\{s\mid s\in S,\ f(s)>\rho\}\in\mathcal{B}$. The set of non-negative $\mathcal{B}$-measurable functions is denoted by $\mathcal{F}^+(\mathcal{B})$; generally, if $\mathcal{A}$ is any set of functions, $\mathcal{A}^+$ denotes the set of non-negative functions in $\mathcal{A}$. The *support* $[f]$ (called "locus" in [4; 5]) of a function f on S is the set $\{s\mid f(s)\neq 0\}$. A function f is "null" or *$\mathcal{N}$-negligible* if $[f]\in\mathcal{N}$; the set of $\mathcal{N}$-negligible functions is denoted by $\mathcal{Z}(\mathcal{N})$. The equivalence class modulo $\mathcal{Z}(\mathcal{N})$ of a function f is written $f+\mathcal{Z}(\mathcal{N})$ or $\bar{f}$ (and not $\{f\}$, as in [2; 4]). We write $\bar{f}\ll\bar{g}$ to mean $f(s)<g(s)$ for all $s\in S-N$ where $N\in\mathcal{N}$, and $\bar{f}\leqq\bar{g}$ to mean $f(s)\leqq g(s)$, $s\in S-N$; $\bar{f}<\bar{g}$ means $\bar{f}\leqq\bar{g}$ and $\bar{f}\neq\bar{g}$. The set $\{\bar{f}\mid f\in\mathcal{F}(\mathcal{B})\}$ is written $\mathcal{F}(\mathcal{B})/\mathcal{Z}(\mathcal{N})$ or $\mathcal{F}(S,\mathcal{B},\mathcal{N})$.(4)

If E is the σ-algebra $\mathcal{B}/\mathcal{N}$, E determines $\mathcal{F}(S,\mathcal{B},\mathcal{N})$ to within "strict" isomorphism (a 1-1 correspondence preserving $\leqq$ and (pointwise) sums and products) (see [4, p. 159]), and we write the strict isomorphism class of $\mathcal{F}(S,\mathcal{B},\mathcal{N})$ as $F(E)$; we also use $F(E)$ to stand for any one member of this class. $F^+(E)$ of course denotes the subset of $F(E)$ corresponding to the non-negative functions. We say that $\mathcal{F}(S,\mathcal{B},\mathcal{N})$ is a *realisation* of $F(E)$. For every σ-algebra E, the function space $F(E)$ exists and has the *representation space realisation* $\mathcal{F}(R,\mathcal{B},\mathcal{N})$ where $R=$representation space of E, $\mathcal{B}=\mathcal{B}R$ (or $\mathcal{B}^*R$ if E is complete), $\mathcal{N}=\mathcal{K}R$. If E is complete, each $\bar{f}\in F(E)$ now has a unique *continuous* representative $f\in\mathcal{F}(R,\mathcal{B},\mathcal{N})$, as follows from [2, p. 285].

The *support* $[\bar{f}]$ of $\bar{f}\in F(E)$ is the element $[f]+\mathcal{N}$ of E. Dually, if $x\in E$, its *characteristic function* χ_x or χx is the element $\overline{\chi(X)}$ of $F(E)$, where $\chi(X)$ is the characteristic function of any set $X\in\mathcal{B}$ for which the equivalence class $X+\mathcal{N}=x$. The value of $\chi(X)$ at $s\in S$ is denoted by $\chi(X,s)$.

2.5 *Cylinder mappings.* Let A be a σ-subalgebra of a σ-algebra E. There is then a natural strict isomorphism c of $F(A)$ in $F(E)$, which we call the "cylinder mapping" (by analogy with the case in which E is the product of A with another factor); it can be described as follows. Each $\bar{g}\in F(A)$ determines a "spectrum" (cf. [6; 4, p. 159]) on A; in terms of any realisation $\mathcal{F}(S,\mathcal{B},\mathcal{N})$ of $F(A)$, the spectrum consists of the equivalence classes modulo $\mathcal{N}$ of the sets $\{s\mid g(s)<\rho\}$ where ρ is rational. Conversely, each spectrum on A determines a unique $\bar{g}\in F(A)$. The imbedding of A in E turns the spectrum of $\bar{g}$ into the spectrum of a unique $\bar{f}\in F(E)$, and we take $c(\bar{g})=\bar{f}$. In particular, if $\mathcal{F}(S',\mathcal{B}',\mathcal{N}')$ is any realisation of $F(E)$, the sets of $\mathcal{B}'$ which correspond to elements in A form a Borel field $\mathcal{B}''\subset\mathcal{B}'$ such that $\mathcal{F}(S',\mathcal{B}'',\mathcal{N}')$ is a realisation of $F(A)$; the cylinder mapping of $F(A)$ in $F(E)$ is now that induced by the identity mapping on S'.

(4) It is an "extended vector σ-lattice with a unit," in the sense that the classes of the finite functions form a vector σ-lattice with a unit, and conversely every vector σ-lattice with a unit arises in this way.

We shall later require the form taken by the cylinder mapping in terms of the representation space realisations. Let R, S be the representation spaces of the σ-algebras E, A where A is a complete subalgebra of E; let ξ be the corresponding mapping of R onto S (cf. 2.2), and let $\mathfrak{C}$, $\mathfrak{D}$ be the families of continuous functions on R, S. As ξ is continuous, ξ induces a mapping ξ^* of $\mathfrak{D}$ in $\mathfrak{C}$ by the rule $(\xi^* g)p = g(\xi p)$, $p \in R$, $g \in \mathfrak{D}$. Each $\bar{g} \in F(A)$ is the equivalence class of a unique $g \in \mathfrak{D}$ (cf. end of 2.4), and we have

$$(1) \qquad c\bar{g} = \overline{\xi^* g},$$

as follows from 2.2(1) and 2.3(2) applied to the spectrum of $\bar{g}$.

2.6 *F'-integrals.* Let E, E' be σ-algebras satisfying the countable chain condition, and write $F = F(E)$, $F' = F(E')$. A mapping ϕ of a subset G of F in F' is called a *positive linear operator from F to F'*, or an *F'-integral on F* (cf. [4, p. 161; 5, p. 232]) if $G \supset F^+$ and:

(1) If $\bar{f} \in F^+$, $\phi\bar{f} \geqq 0$.

(2) If $\bar{f}_n \in F^+$ $(n = 1, 2, \cdots)$, then $\phi(\sum \bar{f}_n) = \sum \phi \bar{f}_n$.

(3) There exist $\bar{g}_1, \bar{g}_2, \cdots \in F^+$ such that $\sum \bar{g}_n \gg 0$ and $\phi \bar{g}_n \ll \infty$.

(4) $G = \{\bar{f} \mid \phi(\bar{f}^+) \wedge \phi(\bar{f}^-) \ll \infty\}$, and if $\bar{f} \in G$ then $\phi\bar{f} = \phi(\bar{f}^+) - \phi(\bar{f}^-)$.

As ϕ is determined by its values on F^+, we shall usually regard ϕ as a mapping of F^+ in F'^+ satisfying (2) and (3); for every such mapping can be extended to a suitable G [4, pp. 161, 162]. The extended mapping ϕ is necessarily linear on G.

If further ϕ satisfies

(i) $\phi\bar{f} > 0$, if $\bar{f} > 0$,

(ii) $\phi 1 \gg 0$,

ϕ is called *strict*. (In [4; 5], a *strict* F'-integral on F was called simply an F'-integral; the present F'-integral was called "relaxed.") Every F'-integral ϕ on F determines in a natural way a strict F_1'-integral ϕ_1 on F_1, where $F_1 = F(E_1)$, $F_1' = F(E_1')$, and E_1, E_1' are suitable principal ideals of E, E' (see [5, p. 238]); ϕ_1 is called the "strict form" of ϕ.

For any F'-integral ϕ on F, we write $\lambda x = \phi(\chi x)$ $(x \in E)$; λ is the induced "F'-measure" on E; it is countably additive and σ-finite [5, p. 233 (a) and (b)] and determines ϕ uniquely.

An F'-integral ϕ on F is *fully homogeneous* if the corresponding F-measure λ is "full-valued" in the sense of [4, p. 166]; that is, given $x \in E$ and $\bar{g} \in F'^+$ such that $\bar{g} \leqq \lambda x$, there exists $y \leqq x$ such that $\lambda y = \bar{g}$. A fully homogeneous ϕ is itself full-valued [4, p. 174]; that is, given $\bar{f} \in F^+$ and $\bar{g} \in F'^+$ such that $\bar{g} \leqq \phi\bar{f}$, there exists $\bar{h} \in F^+$ such that $\bar{h} \leqq \bar{f}$ and $\phi\bar{h} = \bar{g}$.

Let E, E', E'' be σ-algebras, and F, F', F'' their function spaces (that is, $F' = F(E')$, etc.). Suppose ϕ is an F'-integral on F, and ψ an F''-integral on F'. In general, $\psi\phi$ need not be an F''-integral on F, as the σ-finiteness requirement (3) may fail. But:

(5) If ϕ and ψ are fully homogeneous, $\psi\phi$ is a fully homogeneous F''-

integral on F, provided that E satisfies the countable chain condition.

There is no difficulty in seeing that $\psi\phi$ satisfies conditions (1) and (2). Suppose $0 \leqq g \leqq \psi\phi\chi x$ where $x \in E$, $g \in F''$. Put $h = \lambda x = \phi\chi x$; then $0 \leqq g \leqq \psi h$, so (as ψ is full-valued) there exists $k \in F'^{+}$ such that $k \leqq h$ and $\psi k = g$. Hence there exists $y \leqq x$ in E such that $\lambda y = k$; and $\psi\phi\chi y = g$, proving that $\psi\phi$ is fully homogeneous. Put $\psi\phi = \theta$; condition (3) now follows in this way. Put $h_1 = 1 \wedge \theta 1$; there exists $x_1 \in E$ such that $\theta\chi x_1 = h_1$. If α is any countable ordinal, and disjoint elements $x_\beta \in E$ have been defined for all $\beta < \alpha$ so that $\theta\chi x \leqq 1$, we put $h_\alpha = 1 \wedge \theta\chi(-\bigvee x_\beta \mid \beta < \alpha)$. If $h_\alpha \neq 0$, we take $x_\alpha \leqq -\bigvee x_\beta$ so that $\theta\chi x_\alpha = h_\alpha \leqq 1$; if $h_\alpha = 0$, we put $x_\alpha = -\bigvee x_\beta$ and terminate the process. Because of the countable chain condition, this process terminates for some countable α. Renumbering the elements $x_\beta (\beta < \alpha)$ into a simple sequence $x_1, \cdots, x_n, \cdots$, we put $\bar{g}_n = \chi x_n$ and have $\sum \bar{g}_n = 1 \gg 0$ and $\theta \bar{g}_n \leqq 1 \ll \infty$ as required.

For any F'-integral ϕ on F we have:

(6) If $f, g \in F^+$ and $[f] = [g]$, then $[\phi f] = [\phi g]$.

For $\infty \cdot f = \infty \cdot g$; hence $[\phi f] = [\infty \phi f] = [\phi \infty f] = [\phi g]$.

Now let E_1, E_1' be σ-subalgebras of σ-algebras E_2, E_2' satisfying the countable chain condition; write F_i, F_i' for $F(E_i)$, $F(E_i')$ respectively $(i = 1, 2)$, and let c, c' be the respective cylinder mappings of F_1 in F_2, F_1' in F_2'. If ϕ_i $(i = 1, 2)$ is an F_i'-integral on F_i we say that ϕ_2 is a *cylinder extension* of ϕ_1 if, for each $\bar{f} \in F_1^+$, $c'(\phi_1 \bar{f}) = \phi_2(c\bar{f})$. As remarked in 2.5, we can choose realisations of F_i and F_i' for which c' and c are induced by identity mappings, and then ϕ_2 is a cylinder extension of ϕ_1 if and only if ϕ_1 is (in an obvious sense) the restriction of ϕ_2 to F_1.

The following result is basic for the construction in the present paper. It is proved (in a slightly different formulation) in [4, Theorem 6] for the case in which ϕ is strict; the general result follows easily on considering the "strict form" of ϕ. (For details see the beginning of §7.)

(7) If ϕ is an F'-integral on F, where $F = F(E)$ and $F' = F(E')$, there exists an algebra E^*, of which E is a complete subalgebra, and an F'-integral ϕ^* on $F(E)$, such that ϕ^* is a fully homogeneous cylinder extension of ϕ.

Except where the contrary is stated, all algebras in what follows are assumed to be σ-algebras satisfying the countable chain condition. Further, the term "subalgebra" is understood to mean a σ-subalgebra, and hence a complete subalgebra.

3. **Functions on representation spaces.**

3.1 Let A be a (complete) subalgebra of an *arbitrary* algebra E; A is of course assumed to be a σ-algebra satisfying the countable chain condition, but E is not. We derive for later use some properties of the representation spaces R, S of E, A, and of various families of functions on them. We write $\mathfrak{N}_0' = \mathfrak{B}_0 \cap \mathfrak{K}R$, where $\mathfrak{B}_0$ is the Borel field generated by the sets Ry, $y \in A$, and $\mathfrak{N}_0$ for the family of all subsets N of R which are subsets of sets in $\mathfrak{N}_0'$. From 2.2 and 2.3(4), $\mathfrak{F}(R, \mathfrak{B}_0, \mathfrak{N}_0)$ and $\mathfrak{F}(R, \mathfrak{B}_0, \mathfrak{K}R)$ are realisations of $F(A)$; if E

is a σ-algebra, the corresponding cylinder mappings of $F(A)$ in $F(E)$ are induced by the identity mapping. As in 2.2 we write ξ for the natural mapping of R onto S, and $\mathfrak{C}$, $\mathfrak{D}$ for the families of continuous functions on R, S.

3.2 LEMMA. *If $H\in\mathcal{K}S$, there exist $x_{mn}\in A$ $(m, n=1, 2, \cdots)$ such that $x_{m1}\geqq x_{m2}\geqq \cdots$, $\bigwedge_n x_{mn}=0$ for each m, and $H\subset\bigcup_m \bigcap_n Sx_{mn}$.*

For let F be closed and nowhere dense in S; the open set $S-F$ can be written as $\bigcup Sy_\lambda$ for suitable $y_\lambda\in A$, and (because F is nowhere dense) $\bigvee y_\lambda=e$. There is therefore a sequence of values of λ, which we denote by $1, 2, \cdots, i, \cdots$, such that $\bigvee y_i=e$. Put $x_n=-(y_1\vee y_2\vee \cdots \vee y_n)$; then $x_1\geqq x_2\geqq \cdots$, $\bigwedge x_n=o$, and $Sx_n=S-(Sy_1\cup\cdots\cup Sy_n)\supset F$. Now we have $H\subset\bigcup F_m$ $(m=1, 2, \cdots)$ where F_m is closed and nowhere dense; applying the foregoing to F_m instead of F, we obtain the elements x_{mn} required.

COROLLARY. *If $H\in\mathcal{K}S$, there exists $H^*\in\mathcal{K}S\cap\mathfrak{B}S$ such that $H\subset H^*$.*

3.3 DEFINITION. A function f on R is *0-continuous* if, for each real (or, equivalently, rational) ρ, $\{p\mid p\in R, f(p)>\rho\}$ is a union of sets of the form Ry where $y\in A$. (This implies that f is continuous.) The set of 0-continuous functions on R is denoted by $\mathfrak{C}_0$. We have

(1) $f\in\mathfrak{F}(\mathfrak{B}_0)$ if and only if there exists $h\in\mathfrak{F}(\mathfrak{B}(S))$ such that $f=h\xi$.

The nontrivial implication here can be seen by considering the spectrum of f, or by observing that (by the argument in [4, p. 157]) each $f\in\mathfrak{F}(\mathfrak{B}_0)$ is expressible as $\sum\alpha_n\chi X_n$ where $X_n\in\mathfrak{B}_0$ and α_n is real. We have $X_n=\xi^{-1}Y_n$ where $Y_n\in\mathfrak{B}S$, and then, if we set $h=\sum\alpha_n\chi Y_n$, we have $f=h\xi$.

We deduce:

(2) If $f\in\mathfrak{F}(\mathfrak{B}_0)$, there exists $h\in\mathfrak{C}_0$ such that $f=h \bmod \mathfrak{N}_0$.

By (1), $f=g\xi$ where $g\in\mathfrak{F}(\mathfrak{B}(S))$. There exists a continuous function g_1 on S such that $g(s)=g_1(s)$ for $s\in S-H$, where $H\in\mathcal{K}S$; and by 3.2, Corollary, we may assume $H\in\mathfrak{B}S$ also. Put $h=g_1\xi$. Using 2.2(1) we see that h is 0-continuous; and $f(p)=h(p)$ for $p\in R-N$ where $N=\xi^{-1}H\in\mathfrak{N}_0$ by 2.3(4). Conversely:

(3) If $f\in\mathfrak{C}_0$, there exists $g\in\mathfrak{F}(\mathfrak{B}_0)$ such that $f=g \bmod \mathfrak{N}_0$.

For write $X_\rho=\{p\mid p\in R, f(p)<\rho\}$; by hypothesis, this is of the form $\bigcup Ry_\alpha$ for suitable elements $y_\alpha\in A$. Let $z_\rho=\bigvee y_\alpha$; then $Rz_\rho=X_\rho\cup H_\rho$ where $H_\rho=\xi^{-1}(Sz_\rho-\bigcup Sy_\alpha)$. As $Sz_\rho-\bigcup Sy_\alpha$ is closed and nowhere dense in S, it is contained in a set $K_\rho\in\mathcal{K}S\cap\mathfrak{B}S$ (3.2, Corollary); hence $H_\rho\subset\xi^{-1}K_\rho\in\mathfrak{N}_0$. Put $N=\bigcup\{\xi^{-1}K_\rho\mid\rho \text{ rational}\}$, $g=f\chi(R-N)$. Then $N\in\mathfrak{N}_0$, and $f(p)=g(p)$ for $p\in R-N$. Let $Y_\rho=\{p\mid p\in R, g(p)<\rho\}$; one verifies that, if ρ is rational, $Y_\rho=Rz_\rho\cap(R-N)$ for $\rho\leqq 0$, $Y_\rho=Rz_\rho\cup N$ for $\rho>0$. Hence $Y_\rho\in\mathfrak{B}_0$ for all rational ρ, and hence for all ρ, proving $g\in\mathfrak{F}(\mathfrak{B}_0)$.

Next we deduce:

(4) $f\in\mathfrak{C}_0$ if and only if $f=h\xi$ for some $h\in\mathfrak{D}$.

The "if" is trivial from 2.2(1). Conversely, given $f\in\mathfrak{C}_0$, apply (3) and (1) to obtain $g \in \mathfrak{F}(\mathfrak{B}(S))$ such that $f = g\xi$ modulo $\mathfrak{N}_0$. There exists a continuous h on S such that $h(s) = g(s)$ for $s \in S - H$, where $H \in \mathfrak{K}S$. Thus $f(p) = h\xi(p)$ for all $p \in R - N$ where $N \in \mathfrak{K}R$. As $R - N$ is dense in R, and f, $h\xi$ are both continuous, it follows that $f(p) = h\xi(p)$ for all $p \in R$.

3.4 Let $\mathfrak{M}_0$ denote $\mathfrak{B}_0 + \mathfrak{N}_0$ (that is, the family of all sets of the form $B + N$ where $B \in \mathfrak{B}_0$, $N \in \mathfrak{N}_0$). Equivalently (from 2.1(2), 2.2(3), 2.3(4)) $\mathfrak{M}_0$ consists of all sets of the form $Ry + N$ where $y \in A$, $N \in \mathfrak{N}_0$. Clearly $\mathfrak{M}_0$ is a Borel field. We say that a function f on R is *0-measurable* if all the sets $\{p \mid f(p) < \rho\}$ are in $\mathfrak{M}_0$. If $f \in \mathfrak{Z}(\mathfrak{N}_0)$ (that is, if $[f] \in \mathfrak{N}_0$) we say that f is *0-negligible*. Every 0-continuous function is 0-measurable (cf. end of 2.1). Conversely,

(1) Given an 0-measurable function f, there exists a unique 0-continuous function g such that $f = g$ mod $\mathfrak{N}_0$.

By [4, p. 157] we have $f = \sum \alpha_n \chi X_n$ for suitable real numbers α_n and sets $X_n \in \mathfrak{M}_0$ $(n = 1, 2, \cdots)$; and each X_n has the form $Rx_n + H_n$ where $H_n \in \mathfrak{N}_0$ and $x_n \in A$. Consider the function $h = \sum \alpha_n \chi S x_n$ on S; being $\mathfrak{B}S$-measurable, it differs from a continuous function k on S on a first category set K. Then $g = k\xi$ is 0-continuous, and we have $f(p) = k\xi(p)$ if $p \in R - \{\xi^{-1}K \cup \bigcup H_n\}$. The uniqueness of g is trivial.

Given $\bar{g} \in F(A)$, the class $\bar{g}$, in the representation space realisation of $F(A)$, contains a unique continuous function g_0 [2, p. 287]. Then $g_0\xi$ is 0-continuous on R; further, from 3.3(4), every 0-continuous function arises in this way. We write $g_0\xi = R_0\bar{g}$; R_0 induces a strict isomorphism (a 1-1 correspondence preserving $\leqq$ and finite sums and products) between $F(A)$ and $\mathfrak{C}_0$. It follows that every subset $\{f_\alpha\}$ of $\mathfrak{C}_0$ has a least upper bound $f = \bigvee f_\alpha$ in $\mathfrak{C}_0$, and there is a countable subfamily $\{f_{\alpha_n}\}$ of $\{f_\alpha\}$ such that $f = \bigvee f_{\alpha_n}$. Moreover we have

(2) If $f = \bigvee f_\alpha$ in $\mathfrak{C}_0$, $f(p) = \sup f_\alpha(p)$ for all $p \in R - N$, where $N \in \mathfrak{N}_0$.

For we have $f_\alpha = g_\alpha \xi$ where g_α is continuous on S, and if $\bar{g} = \bigvee \bar{g}_\alpha$ in $F(A)$ then $\bar{g} = \bigvee \bar{g}_{\alpha_n}$. If g_0 is the continuous function on S which is in $\bar{g}$, then $g_0(s) = \sup g_{\alpha_n}(s)$ if $s \in S - H$ where $H \in \mathfrak{K}S$, and we can assume (3.2) $H \in \mathfrak{B}S$ also. Then $f = g_0\xi$, and $f(p) = \sup f_\alpha(p)$ if $p \in R - \xi^{-1}H$ where $\xi^{-1}H \in \mathfrak{N}_0$ by 2.3(4).

Note that, from 2.2(1),

(3) $$R_0 \chi y = \chi R y, \qquad \text{if } y \in A.$$

If E is itself a σ-algebra, the cylinder mapping c of $F(A)$ in $F(E)$ is defined, and from 2.5(1) we have, for $\bar{g} \in F(A)$,

(4) $R_0\bar{g}$ is the unique 0-continuous function in the class $c\bar{g}$.[5]

For later use, we deduce:

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[5] Even when E is only finitely additive, it would be possible to define a "cylinder mapping" from $F(A)$ to the finitely additive function space corresponding to $F(E)$, whenever A is a σ-subalgebra of E. This mapping would then be realised by R_0.

(5) If $f \in \mathfrak{C}_0^+$, there exist *positive* real numbers σ_n, elements $x_n \in A$ ($n=1, 2, \cdots$), and a *non-negative* function h such that

$$f = \sum \sigma_n \chi R x_n + h.$$

We have $f = R_0 \bar{g}$ where $\bar{g} \in F(A)^+$, and a slight modification of the argument in [4, p. 157, Lemma 1] gives $\bar{g} = \sum \sigma_n \chi x_n$ where $x_n \in A$ and $\sigma_n > 0$. Hence, by (3), $f = \sum \sigma_n \chi R x_n$ on $R - N$, where $N \in \mathfrak{N}_0$. By continuity, $f \geqq$ every finite sum of terms $\sigma_n \chi R x_n$, so the difference h between f and $\sum \sigma_n \chi R x_n$ is $\geqq 0$.

4. The main theorem and its proof (first part).

4.1 We now state the main theorem of this paper. We recall that "algebra" means "Boolean σ-algebra satisfying the countable chain condition" except where the contrary is stated.

THEOREM 1. *Let E_0 be an algebra, $F_0 = F(E_0)$ its function space, and ϕ_0 a positive linear operator from F_0 to itself (that is, an F_0-integral on F_0). There exist an algebra E, of which E_0 is a subalgebra, and a fully homogeneous F-integral ϕ on F, where $F = F(E)$, such that ϕ is a cylinder extension of ϕ_0.*

REMARK. If ϕ_0 is a *strict* F_0-integral on F_0, I do not know whether ϕ can always be taken to be a *strict* F-integral on F. If $\phi_0(1) \gg 0$, then automatically $\phi(1) \gg 0$ because ϕ is a cylinder extension.

Before proving the theorem, we note the following consequence of it.

COROLLARY. *For each $n=1, 2, \cdots$, ϕ^n is also a fully homogeneous F-integral on F, and is a cylinder extension of ϕ_0^n.*

This follows from Theorem 1 by an easy induction, using 2.6(5).

4.2 *The algebras E_n.* The proof of Theorem 1 requires a number of steps. First we note that by successive applications of 2.6(7) we obtain a sequence of algebras $E_0, E_1, E_2, \cdots$, where E_k is a subalgebra of E_{k+1}, and a sequence $\{\phi_k\}$ ($k=0, 1, 2, \cdots$) where ϕ_k is an F_k-integral on F_k, F_k denoting $F(E_k)$, such that

(1) ϕ_{k+1} is a cylinder extension of ϕ_k; that is, $\phi_{k+1} c_{k,k+1} = c_{k,k+1} \phi_k$, where $c_{k,k+1}$ is the cylinder mapping of F_k in F_{k+1},

(2) $\phi_{k+1}(F_{k+1}^+) \subset c_{k,k+1}(F_k^+)$, and further

(3) $(c_{k,k+1})^{-1} \phi_{k+1}$ is a fully homogeneous F_k-integral on F_{k+1}.

(We merely put $\phi_{k+1} = c_{k,k+1} \phi^*$ where ϕ^* is the extension provided by 2.6(7).)

We write $c_{n,n+k}$ for the cylinder mapping of F_n into F_{n+k} ($n, k \geqq 0$), noting that $c_{n,n+k}$ is 1-1, that c_{nn} is the identity mapping, and that $c_{n,n+h+k} = c_{n+h,n+h+k} c_{n,n+h}$. To simplify printing, we write the inverse mapping $(c_{n,n+k})^{-1}$ as $c_{n+k,n}$. By induction, first over k for $m=1$ and then over m, we obtain

$$(4) \qquad (\phi_{n+k})^m c_{n,n+k} = c_{n,n+k} (\phi_n)^m \text{ on } F_n^+ \qquad (m = 1, 2, \cdots).$$

Restated in terms of the inverse cylinder mappings, this is

(5) $$c_{n+k,n}(\phi_{n+k})^m = (\phi_n)^m c_{n+k,n} \text{ on } c_{n,n+k}\overset{+}{F_n} \subset \overset{+}{F}_{n+k}.$$

Next we show

(6) $$(\phi_{n+k})^k \overset{+}{F}_{n+k} \subset c_{n,n+k}\overset{+}{F_n}, \quad \text{and} \quad c_{n+k,n}(\phi_{n+k})^k \text{ is a fully homogeneous } F_n\text{-integral on } F_{n+k}.$$

In fact, if $r \geqq 1$, $c_{n+r,n+r-1}\phi_{n+r}$ is a fully homogeneous F_{n+r-1}-integral on F_{n+r}, by (3). Now, on $\overset{+}{F}_{n+k}$, put

$$\psi = (c_{n+1,n}\phi_{n+1})(c_{n+2,n+1}\phi_{n+2}) \cdots (c_{n+k,n+k-1}\phi_{n+k}).$$

Then ψ is a fully homogeneous F_n-integral on F_{n+k}, by 2.6(5). But, in view of (5),

$$\begin{aligned} \psi &= c_{n+1,n}(\phi_{n+1}c_{n+2,n+1}) \cdots (\phi_{n+k-1}c_{n+k,n+k-1})\phi_{n+k} \\ &= c_{n+1,n}c_{n+2,n+1}\phi_{n+2}\phi_{n+2}c_{n+3,n+2} \cdots \\ &= c_{n+2,n}(\phi_{n+2})^2 c_{n+3,n+2}\phi_{n+3} \cdots \\ &= c_{n+2,n}c_{n+3,n+2}(\phi_{n+3})^3 \cdots = c_{n+k,n}(\phi_{n+k})^k \text{ finally.} \end{aligned}$$

This also shows $c_{n+k,n}(\phi_{n+k})^k$ is defined for all $\bar{f} \in \overset{+}{F}_{n+k}$, giving the first part of the assertion.

REMARK. It follows from (6) that $(\phi_n)^m$ is an F_n-integral on F_n provided $m \leqq n$; compare footnote 3.

4.3 *The algebra E'*. Let $E' = \bigcup E_n$, where $E_0, E_1, E_2, \cdots$ is the sequence of algebras obtained in 4.2. For $x, y \in E'$, define $x \leqq y$ to mean that $x \leqq y$ in some E_n (and so in E_m for all $m \geqq n$). It is easily verified that E' becomes a *finitely* additive Boolean algebra satisfying the countable chain condition, and that each E_n is a complete subalgebra of E'. Let R be the representation space of E'. The required extended function space F of Theorem 1 (4.1) will be defined by a certain realisation $\mathfrak{F}(R, \mathfrak{B}, \mathfrak{N})$; but before we define $\mathfrak{B}$ and $\mathfrak{N}$ it is convenient to have the extended integral ϕ more or less available. This we achieve by defining an operator Φ on a suitable class $\mathfrak{C}'$ of continuous functions on R (4.5). By measure-theoretic considerations we are then able to extend a modified form of Φ to an operator Φ^* on the family $\mathfrak{F}(\mathfrak{M}')$ of $\mathfrak{M}'$-measurable functions, where $\mathfrak{M}'$ is a certain Borel field of subsets of R (4.8); and all that remains is to define the ideal $\mathfrak{N}$ of null sets—an operation of some delicacy since $\mathfrak{N}$ must be large enough to produce the countable chain condition and not so large as to annihilate Φ^*.

4.4 *The function-class $\mathfrak{C}'$*. As in §2, we let Rx denote the open-closed subset of R corresponding to $x \in E'$. We write $\mathcal{E}' = \{Rx \mid x \in E'\}$, $\mathcal{E}_k = \{Rx \mid x \in E_k\}$, $\mathfrak{B}' = \mathfrak{B}R =$ Borel field (of subsets of R) generated by $\mathcal{E}'$, $\mathfrak{B}_k =$ Borel field generated by $\mathcal{E}_k$. A set $N \subset R$ is called *k-negligible* if it is contained in some $Y \in \mathfrak{B}_k \cap \mathcal{K}R$; N is *negligible* if it is of the form $\bigcup N_k$ $(k = 0, 1, \cdots)$ where N_k is k-negligible. The families of k-negligible and of negligible sets are written

$\mathfrak{N}_k$, $\mathfrak{N}'$ respectively; they are σ-ideals. We have $\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \cdots \subset \mathfrak{B}'$, $\mathfrak{N}_0 \subset \mathfrak{N}_1 \subset \cdots \subset \mathfrak{N}' \subset \mathfrak{K}R$. We say that a function on R is *k-negligible* or *negligible* if it is in $\mathfrak{Z}(\mathfrak{N}_k)$ or $\mathfrak{Z}(\mathfrak{N}')$, respectively.

We define $\mathfrak{M}_k = \mathfrak{B}_k + \mathfrak{N}_k$ (cf. 3.4), $\mathfrak{M}_k' = \mathfrak{B}_k + \mathfrak{N}'$, $\mathfrak{M}' = \mathfrak{B}' + \mathfrak{N}'$; all these are Borel fields. By 3.3(2), $\mathfrak{B}_k \subset \mathfrak{E}_k + \mathfrak{N}_k$, so that $\mathfrak{M}_k = \mathfrak{E}_k + \mathfrak{N}_k$, and hence $\mathfrak{M}_k' = \mathfrak{E}_k + \mathfrak{N}'$. Clearly $\mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots$, $\mathfrak{M}_0' \subset \mathfrak{M}_1' \subset \cdots$, and $\mathfrak{M}_k \subset \mathfrak{M}_k' \subset \mathfrak{M}'$. If $x \in E_k$, the correspondence between x and $(Rx) + \mathfrak{N}_k$ is a (complete) isomorphism between E_k and $\mathfrak{M}_k/\mathfrak{N}_k$; similarly, for $x \in E'$, the correspondence between x and $(Rx) + \mathfrak{N}'$ is a *finite* isomorphism between E' and a *finitely* additive subalgebra of $\mathfrak{M}'/\mathfrak{N}'$. (Note that in general E' need not be a σ-algebra, and that the σ-algebra $\mathfrak{M}'/\mathfrak{N}'$ need not satisfy the countable chain condition.) If we restrict x to E_k here, we obtain an isomorphism of E_k onto the (complete) subalgebra $\mathfrak{M}_k'/\mathfrak{N}'$ of $\mathfrak{M}'/\mathfrak{N}'$.

We call a function f on R "k-continuous" if for each real ρ $\{p \mid f(p) > \rho\}$ is a union of sets in $\mathfrak{E}_k$; that is, if f is "0-continuous" in the sense of 3.3, taking $E = E'$, $A = E_k$. We write $\mathfrak{C}_k$ for the family of all k-continuous functions on R, and $\mathfrak{C}'$ for $\bigcup \mathfrak{C}_k$; note that $\mathfrak{C}_0 \subset \mathfrak{C}_1 \subset \cdots$. A function is called "$k$-measurable" if it is in $\mathfrak{F}(\mathfrak{M}_k)$ (this notation is consistent with that in 3.4), "k'-measurable" if it is in $\mathfrak{F}(\mathfrak{M}_k')$. The following assertions follow easily from 3.3(3) and 3.4(1):

(1) Every k-continuous function is k-measurable, and hence k'-measurable.

(2) If f is k-measurable, there is a unique k-continuous function g_k such that $f = g_k$ mod $\mathfrak{N}_k$; and $g_k = g_{k+1} = \cdots$.

Let R_k be the isomorphism between $F_k = F(E_k)$ and $\mathfrak{C}_k$ described in 3.4 (where we replace A by E_k, E by E' and R_0 by R_k).(6) If we use the realisation $\mathfrak{F}(R, \mathfrak{M}_k', \mathfrak{N}')$ of F_k, a typical element $\bar{f}$ of F_k consists of all functions differing from a k'-measurable function f by a negligible function, the cylinder mapping $c_{k,k+n}$ becomes the identity mapping, and $R_k\bar{f}$ is the unique k-continuous function in $\bar{f}$. It follows that, for arbitrary realisations,

$$R_k\bar{f} = R_{k+n}c_{k,k+n}\bar{f} \qquad (k, n = 0, 1, 2, \cdots, \bar{f} \in F_k). \tag{3}$$

4.5 *The operator* Φ. Given $f \in \mathfrak{C}'^+$, we have $f \in \mathfrak{C}_k$ for some k; put $\bar{g} = R_k^{-1}f \in F_k$, and define $\Phi f = R_k\phi_k\bar{g} \in \mathfrak{C}_k^+$. This definition does not depend on the choice of k, as follows from 4.4(3) and 4.2(4), so Φ is a single-valued mapping of $\mathfrak{C}'^+$ into itself. Φ and its iterates have the following properties; in all of them, $k = 0, 1, 2, \cdots, m, n = 1, 2, \cdots$, and, where no proofs are given, the proofs are straightforward inductions over m.

(1) If $f = R_k\bar{g}$, where $\bar{g} \in F_k^+$, then $\Phi^m f = R_k\phi_k^m\bar{g}$.

(2) $\Phi^m(C_{k+m}^+) \subset C_k^+$.

The case $m = 1$ of (2) follows from the definition of Φ, together with 4.2(2) and 4.4(3); the general case then follows by induction over m.

(6) Our notation in this paragraph is not quite exact; some isomorphisms have been suppressed. Strictly speaking, R_k depends on which realisation of F_k is used, but this should not cause confusion here.

(3) If f, $g\in\mathcal{C}'^{+}$ and a, b are non-negative real numbers,

$$\Phi^m(af + bg) = a\Phi^m f + b\Phi^m g.$$

(4) Let f_n, $f\in\mathcal{C}_k^+$, and write $\Phi^m f_n = g_n$, $\Phi^m f = g$. Suppose that $f_n(p)\to f(p)$ for each $p\in R-N$, where $N\in\mathfrak{N}_k$, and that either (a) $f_1\leqq f_2\leqq\cdots$ or (b) $f_1\geqq f_2\geqq\cdots$ and $g_1(p)<\infty$ if $p\in R-N$. Then $g_n(p)\to g(p)$ for all $p\in R-N^*$ where $N^*\in\mathfrak{N}_k$.

(5) If $X\in\mathcal{E}_{k+m}$, $g\in\mathcal{C}_k^+$, and $g\leqq\Phi^m\chi X$, then $g=\Phi^m\chi Y$ for some $Y\in\mathcal{E}_{k+m}$.

Here induction is not needed for the proof, which is a straightforward calculation based on 4.2(6).

As an immediate consequence of (5), Φ^m is "fully homogeneous" in the following sense:

(6) *If* $X\in\mathcal{E}'$, $g\in\mathcal{C}'^{+}$, *and* $g\leqq\Phi^m\chi X$, *then* $g=\Phi^m\chi Y$ *for some* $Y\in\mathcal{E}'$.

Finally Φ^m has the following σ-finiteness property:

(7) *Given* k $(=0, 1, 2, \cdots)$, *there exist disjoint sets* $G_n\in\mathcal{E}_{k+1}$ $(n=1, 2, \cdots)$ *such that* $R-\bigcup G_n\in\mathfrak{N}_{k+1}$ *and* $\Phi^k\chi G_n\leqq 1$.

For, by 4.2(6), $c_{k+1,1}(\phi_{k+1})^k$ is a fully homogeneous F_1-integral on F_{k+1}. The argument proving 2.6(5) shows that elements $y_1, y_2, \cdots$ exist in E_{k+1} such that $\bigvee y_n = e$ and $\phi_{k+1}^k\chi y_n\leqq 1$. We take $G_n = Ry_n$; then $G_n\in\mathcal{E}_{k+1}$, $R-\bigcup G_n$ is $(k+1)$-negligible, and finally, by (1) above, $\Phi^k\chi G_n = R_{k+1}\phi_{k+1}^k\chi y_n\leqq 1$.

4.6 *The measures* μ_p *and* ν_p. Using 4.5(7), we take a sequence of disjoint sets $G_n\in\mathcal{E}_1$ such that $R-\bigcup G_n = Z\in\mathfrak{N}_1$ and $\Phi\chi G_n\leqq 1$; these sets will remain fixed throughout the rest of the proof. Let p be any point of R, fixed for the moment. For each $X\in\mathcal{E}'$, put $\mu_p(X)$ = value at p of the function $\sum_n\Phi\chi(X\cap G_n)$. From 4.5(3), μ_p is a finitely additive (non-negative) measure on $\mathcal{E}'$, and $\mu_p(G_n)\leqq 1$. Further, if $X_1, X_2, \cdots$ is any sequence of disjoint sets in $\mathcal{E}'$, and if $X=\bigcup X_n\in\mathcal{E}'$, then $\mu_p(X)=\sum\mu_p(X_n)$ because X is compact and each X_n is open, so that all but a finite number of the sets X_n must be empty. There is therefore [1, p. 2] an extension of μ_p to a complete countably additive measure μ_p' on a Borel field containing $\mathcal{B}\mathcal{E}'=\mathcal{B}'$.(7) For $X\in\mathcal{B}'$ define $\nu_p(X) = \sum_n\mu_p'(X\cap G_n) = \mu_p'(X-Z)$. Then ν_p is a countably additive σ-finite measure on $\mathcal{B}'$, which extends μ_p. We assert:

(1) *Given* $X\in\mathcal{E}'$, $\nu_p(X)=(\Phi\chi X)(p)$ *for each* $p\in R-N_X$, where $N_X\in\mathfrak{N}'$.

For say $X\in\mathcal{E}_k$; then $G_n\in\mathcal{E}_1\subset\mathcal{E}_k$ (we may assume $k>0$), so $X\cap G_n\in\mathcal{E}_k$ and $\nu_p(X)=\sum_n\mu_p(X\cap G_n)=\sum\{\text{value at } p \text{ of } \Phi\chi(X\cap G_n)\}$. Now each $\chi(X\cap G_n)\in\mathcal{C}_k$, and also $\chi(X)\in\mathcal{C}_k$ and $\sum\chi(X\cap G_n)=\chi X$ on $R-Z$ where $Z\in\mathfrak{N}_k$. Hence by 4.5(4) we have $\Phi(\sum\chi(X\cap G_n), 1\leqq n\leqq m)\to\Phi\chi X$ on $R-N_X$ for some $N_X\in\mathfrak{N}_k\subset\mathfrak{N}'$, and the assertion follows.

4.7 *The outer measure function* ν^*. For arbitrary $X\subset R$ and $p\in R$, let ν_p^*X denote the outer measure of X with respect to the measure ν_p. We write ν^*X for the function on R whose value at p is ν_p^*X. If $X\in\mathcal{B}'$, X is ν_p-measurable for every $p\in R$. Generally, if X is ν_p-measurable except for a negligible set of p's, we write νX instead of ν^*X.

(7) A "complete" measure is one for which all subsets of null sets are measurable.

(1) *If* $X \in \mathfrak{N}'$, νX *exists and is negligible.*

We have $X = \bigcup X_k$ where $X_k \in \mathfrak{N}_k$, and we may suppose $k \geqq 1$. As the sets G_n are disjoint and ν_p-measurable for every p, we have $\nu^* X_k = \sum_n \nu^*(X_k \cap G_n)$, so it is enough to prove that $\nu^*(X_k \cap G_n)$ is k-negligible. By 3.2, $X_k \subset \bigcup_m \bigcap_h Rx_{mh}$ where $x_{mh} \in E_k$ and, for fixed m, $x_{m1} \geqq x_{m2} \geqq \cdots$ and $\bigwedge_h x_{mh} = 0$. We have $G_n = Ry_n$ where $y_n \in E_1 \subset E_k$. Let $Z_{mn} = \bigcap_h Rx_{mh} \cap G_n = \bigcap_h R(x_{mh} \wedge y_n)$. The sequence $\{\chi R(x_{mh} \wedge y_n)\}$ $(h = 1, 2, \cdots)$ of functions of $\mathfrak{C}_k^+$ decreases monotonically, and its limit is 0 outside a k-negligible set; further, $\Phi\chi R(x_{m1} \wedge y_n) \leqq \Phi\chi G_n \leqq 1$. Hence, by 4.5(4), $\nu R(x_{mh} \wedge y_n) = \Phi\chi R(x_{mh} \wedge y_n) \to 0$ except on a k-negligible set as $h \to \infty$, proving $\nu^* Z_{mn}$ is k-negligible. As $X_k \cap G_n \subset \bigcup_m Z_{mn}$, this proves $\nu^*(X_k \cap G_n)$ k-negligible, as required.

(2) *If* $X \in \mathfrak{M}'$, X *is* ν_p*-measurable for all* $p \in R - N_X$, *where* $N_X \in \mathfrak{N}'$; *and* $\nu X \in \mathfrak{F}(\mathfrak{M}')$.

First suppose $X \in \mathcal{E}'$. Then X is ν_p-measurable for all $p \in R$. Again, write $X_n = X \cap G_n$; we have $\nu X = \sum \nu X_n = \sum \Phi\chi X_n$ where $\Phi\chi X_n \in \mathfrak{C}' \subset \mathfrak{F}(\mathfrak{M}')$, from 3.3(3), showing that $\nu X \in \mathfrak{F}(\mathfrak{M}')$.

Now suppose $X \in \mathfrak{B}'$. Again X is ν_p-measurable for all $p \in R$, and as above it is enough to prove that each $\nu(X \cap G_n) \in \mathfrak{F}(\mathfrak{M}')$. Thus we may assume $X \subset G_n$. The $\mathfrak{M}'$-measurability of νX now follows by transfinite induction over the rank α of X considered as in the Borel field generated by sets in $\mathcal{E}'$ which are subsets of G_n; we use the facts that X is a limit of a monotone sequence of sets of smaller rank and of finite measure, and that a (pointwise) limit of a sequence of functions in $\mathfrak{F}(\mathfrak{M}')$ is in $\mathfrak{F}(\mathfrak{M}')$.

Finally, if $X \in \mathfrak{M}'$, $X = Y + Z'$ where $Y \in \mathfrak{B}'$, $Z' \in \mathfrak{N}'$; if $p \in R - N_X$ where N_x is negligible, $\nu_p^*(Z') = 0$ by (1), and X is ν_p-measurable and $\nu_p(X) = \nu_p(Y)$. Thus $\nu(X) = \nu(Y) \in \mathfrak{F}(\mathfrak{M}')$.

4.8 *The operator* Φ^*. As a corollary to the last result, we have

(1) If $f \in \mathfrak{F}(\mathfrak{M}')^+$, then f is ν_p-measurable except for a negligible set of p's.

We define $\Phi_p^* f = \inf\{\int_R g d\nu_p | g$ is ν_p-measurable and $g \geqq f\}$. The function on R whose value at p is $\Phi_p^* f$ is denoted by $\Phi^* f$. It is easy to verify that, for arbitrary $X \subset R$,

$$\Phi^* \chi X = \nu^* X. \tag{2}$$

We deduce

(3) *If* $f \in \mathfrak{F}(\mathfrak{M}')^+$, *then* $\Phi^* f \in \mathfrak{F}(\mathfrak{M}')^+$.

For, by a familiar argument, $f = \sum \alpha_n \chi X_n$ where $\alpha_n > 0$, $X_n \in \mathfrak{M}'$. By 4.7(2), X_n is ν_p-measurable for all $p \in R - N_n$ where $N_n \in \mathfrak{N}'$. Put $N = \bigcup N_n$; then, if $p \in R - N$, f is ν_p-measurable and consequently $\Phi_p^* f = \int_R f d\nu_p = \sum \alpha_n \nu_p(X_n)$ where, for each n, the function $\nu_p(X_n)$ of p is in $\mathfrak{F}(\mathfrak{M}')^+$ by 4.7(2). Thus $\Phi^* f$ differs from an $\mathfrak{M}'$-measurable function at most on N, and is therefore $\mathfrak{M}'$-measurable.

(4) *If* $f \geqq 0$ *and* $[f] \in \mathfrak{N}'$, *then* $[\Phi^* f] \in \mathfrak{N}'$.

Let $[f] = X$. By 4.7(1), $\nu_p^* X = 0$ for all $p \in R - N$ where $N \in \mathfrak{N}'$. If $p \in R - N$, we may take $g=\infty\chi X$ in the definition of $\Phi_p^* f$, showing $\Phi_p^* f=0$ for $p\in R-N$.

(5) If $f_n\in\mathfrak{F}(\mathfrak{M}')^+$ $(n=1, 2, \cdots)$, then $\Phi_p^*(\sum f_n)=\sum\Phi_p^* f_n$ except for a negligible set of p's.

The proof resembles that of (3). We immediately deduce:

(6) If $f_n\in\mathfrak{F}(\mathfrak{M}')^+$ $(n=1, 2, \cdots)$, $f_1\geqq f_2\geqq\cdots$, and $\Phi_p^* f_1<\infty$ except on a negligible set, then $\Phi_p^*(\lim f_n)=\lim\Phi_p^* f_n$ except on a negligible set.

If $f\in\mathcal{C}'^+$, the value of Φf at $p\in R$ is denoted by $\Phi_p f$; similarly we define $\Phi_p^m f$.

(7) *If $f\in\mathcal{C}'^+$, then $\Phi_p^* f=\Phi_p f$ if $p\in R-N$ where $N\in\mathfrak{N}'$.*

Say $f\in\mathcal{C}_k$. By 3.4(5) we may write $f=\sum\sigma_n\chi X_n+g$ where $\sigma_n>0$, $X_n\in\mathcal{E}_k$, $g\geqq 0$ and $[g]\in\mathfrak{N}_k\subset\mathfrak{N}'$. From 4.5(4), if $p\in R-N_1$ where $N_1\in\mathfrak{N}_k$, $\Phi_p f=\sum\Phi_p(\sigma_n\chi X_n)=\sum\sigma_n\nu_p(X_n)$ by 4.6, if $p\notin N_2$, where $N_2\in\mathfrak{N}_k$. On the other hand, from (4), $\Phi_p^* g=0$ except on $N_3\in\mathfrak{N}'$, and outside N_3 we have that f is ν_p-measurable and consequently $\Phi_p^* f=\sum\sigma_n\int\chi X_n d\nu_p=\sum\sigma_n\nu_p(X_n)$. Thus $\Phi_p^* f=\Phi_p f$ if $p\in R-(N_1\cup N_2\cup N_3)$.

Since Φ^* maps $\mathfrak{F}(\mathfrak{M}')^+$ in itself, the iterates Φ^{*m} $(m=1, 2, \cdots)$ are all defined; it is easy to see that properties (3)–(6) apply to Φ^{*m}, and similarly (7) gives (with a little more trouble):

(7′) If $f\in\mathcal{C}'^+$, $\Phi^{*m}f$ and $\Phi^m f$ differ only on a negligible set.

4.9 *Support properties of* Φ^*. We list the following properties of Φ^* for later use; they all follow easily from the foregoing. Throughout, it is assumed that $f, f_1, f_2, \cdots\in\mathfrak{F}(\mathfrak{M}')^+$.

(1) $[\Phi^{*m}f]\in\mathfrak{M}'$.

(2) If $[f]\in\mathfrak{N}'$, $[\Phi^{*m}f]\in\mathfrak{N}'$.

(3) $\bigcup_n[\Phi^{*m}f_n]\subset[\Phi^{*m}\sup f_n]\subset[\Phi^{*m}\sum f_n]$, and

$$[\Phi^{*m}\sum f_n]-\bigcup_n[\Phi^{*m}f_n]\in\mathfrak{N}'.$$

(4) If $[f_1]+[f_2]\in\mathfrak{N}'$, then $[\Phi^{*m}f_1]+[\Phi^{*m}f_2]\in\mathfrak{N}'$.

In particular,

$$[\Phi^{*m}f]+[\Phi^{*m}\chi[f]]\in\mathfrak{N}'. \tag{5}$$

4.10 *The set-operator I.* As a preliminary to defining the final ideal $\mathfrak{N}$ of "null sets," we define $I(X)$, for $X\in\mathfrak{M}'$, by: $I(X)=[\sum\Phi^{*m}\chi X]$, where $m=0, 1, 2, \cdots$. Taking $m=0$ shows

$$I(X)\supset X. \tag{1}$$

The following results follow easily with the aid of 4.9. It is assumed throughout that $X, X_1, X_2, \cdots\in\mathfrak{M}'$.

(2) $I(X)\in\mathfrak{M}'$.

(3) If $X\in\mathfrak{N}'$, $I(X)\in\mathfrak{N}'$.

(4) $I(X)=\bigcup[\Phi^{*m}\chi X](m\geqq 0)$.

(5) If $X\subset Y$, $I(X)\subset I(Y)$.

(6) $I(\bigcup X_n)=\bigcup I(X_n)\cup N$, where $N\in\mathfrak{N}'$.

(7) $I(X+Y)\supset I(X)+I(Y)$ mod $\mathfrak{N}'$; hence if $X+Y\in\mathfrak{N}'$,

$$I(X)+I(Y)\in\mathfrak{N}'.$$

(8) $I(I(X))=I(X)$ mod $\mathfrak{N}'$.

(9) If $f\in\mathfrak{F}(\mathfrak{M}')^+$, then $[\sum\Phi^{*m}f]=I[f]$ mod $\mathfrak{N}'$, and hence

$$I[\Phi^*f]\subset I[f] \text{ mod } \mathfrak{N}'.$$

As the last four of these statements are less trivial than the others, we sketch their proofs.

Proof of (6). By (4) and 4.9(3),

$$I(\bigcup X_n) = \bigcup_m [\Phi^{*m}\chi \bigcup X_n] \subset \bigcup_m [\Phi^{*m}\sum \chi X_n] = \bigcup_m \left\{ \bigcup_n [\Phi^{*m}\chi X_n] \cup Z_m \right\}$$

where $Z_m\in\mathfrak{N}'$, $=\bigcup_{m,n}[\Phi^{*m}\chi X_n]\cup Z'=\bigcup I(X_n)\cup Z'$ where $Z'\in\mathfrak{N}'$. But $I(\bigcup X_n)\supset I(X_n)$, by (5).

Proof of (7). By (6), $I(X)=I(X\cap Y)\cup I(X-Y)\cup N_1$, $I(Y)=I(X\cap Y)\cup I(Y-X)\cup N_2$, where N_1, $N_2\in\mathfrak{N}'$; so $I(X)+I(Y)\subset I(X-Y)\cup I(Y-X)\cup N_1\cup N_2\subset I(X+Y)\cup(N_1\cup N_2)$ by (5).

Proof of (8). Using (4), (6) and 4.9(5), we find

$$I(I(X)) = \bigcup_m \bigcup_n [\Phi^{*n}\chi[\Phi^{*m}\chi(X)]] = \bigcup_{m,n} [\Phi^{*m+n}\chi X] \text{ mod } \mathfrak{N}' = I(X) \text{ mod } \mathfrak{N}'.$$

Proof of (9). From (4) and 4.9(5), $I[f]=[\Phi^{*m}\chi[f]]=\bigcup[\Phi^{*m}f]$ mod $\mathfrak{N}'$ $=[\sum\Phi^{*m}f]$. Hence $[\Phi^*f]=I[f]$ mod $\mathfrak{N}'$, giving (from (3) and (8)) $I[\Phi^*f]\subset I(I[f])$ mod $\mathfrak{N}'=I[f]$ mod $\mathfrak{N}'$.

5. **Proof of Theorem 1 concluded.**

5.1 *The σ-ideal* $\mathfrak{N}$. Define $\mathfrak{N}=\{X\mid$ there exists $Y\in\mathfrak{M}'$ such that $X\subset Y$ and $I(Y)\in\mathfrak{K}R\}$. We have at once:

(1) $\mathfrak{N}$ is a σ-ideal. (From 4.10(6).)

(2) $\mathfrak{N}'\subset\mathfrak{N}\subset\mathfrak{K}R$. (From 4.10(3) and 4.10(1).)

(3) If $X\in\mathcal{E}'\cap\mathfrak{N}$, X is empty. (From (2).)

(4) If $f\in\mathfrak{F}(\mathfrak{M}')^+$ and $[f]\in\mathfrak{N}$, then $[\Phi^*f]\in\mathfrak{N}$. (From 4.10.)

(5) If f, $g\in\mathfrak{F}(\mathfrak{M}')^+$ and $[f]+[g]\in\mathfrak{N}$, then $[\Phi^*f]+[\Phi^*g]\in\mathfrak{N}$.

For let $R-X=[f]\cap[g]$, and put $f=f\chi X+f'$, $g=g\chi X+g'$. Then $[f']=[g']$, so $[\Phi^*f']=[\Phi^*g']$ mod $\mathfrak{N}'$, by 4.9(4). Also $\Phi^*f=\Phi^*f'+\Phi^*f\chi X$ mod $\mathfrak{N}'$ $=\Phi^*f'$ mod $\mathfrak{N}$ by (4) and (2). Thus, modulo $\mathfrak{N}$, $[\Phi^*f]=[\Phi^*f']=[\Phi^*g']=[\Phi^*g]$.

5.2 *The algebra E*. Now put $\mathfrak{B}=\mathfrak{M}'+\mathfrak{N}$; this is a Borel field containing $\mathfrak{N}$. Define $E=\mathfrak{B}/\mathfrak{N}$, a Boolean σ-algebra. Since $\mathfrak{M}'=\mathfrak{B}'+\mathfrak{N}'$ and $\mathfrak{N}\supset\mathfrak{N}'$, we have $\mathfrak{B}=\mathfrak{B}'+\mathfrak{N}$, and a typical element of E is thus the class of sets $(X)+\mathfrak{N}$ $(=\{X+N\mid N\in\mathfrak{N}\})$ where $X\in\mathfrak{B}'=\mathfrak{B}R$. We now prove

(1) *E satisfies the countable chain condition.*

Suppose $\mathfrak{A}$ is an uncountable family of sets $A\in\mathfrak{B}'$, none of which is in $\mathfrak{N}$, but such that whenever A_1, A_2 are distinct members of $\mathfrak{A}$ then $A_1 \cap A_2 \in \mathfrak{N}$; we must derive a contradiction. We may suppose that $\mathfrak{A}$ consists of just $\aleph_1$ sets; well-order $\mathfrak{A}$ as $\{A_\alpha | \alpha < \omega_1\}$, and let $A'_\alpha = A_\alpha - \bigcup\{A_\beta | \beta < \alpha\}$; then $A'_\alpha \in \mathfrak{B}'$, $A_\alpha - A'_\alpha \in \mathfrak{N}$, and distinct sets A'_α are disjoint. Thus, replacing $\mathfrak{A}$ by $\{A'_\alpha | \alpha < \omega_1\}$, we may further assume that $\mathfrak{A}$ consists of *disjoint* sets.

For each $A \in \mathfrak{A}$, there is a least $n \geqq 0$ such that $[\Phi^{*n}\chi A]$ is of second category in R (else $I(A) \in \mathcal{K}R$ and $A \in \mathfrak{N}$). Let $\mathfrak{A}_k$ be the subfamily of $\mathfrak{A}$ for which this n has the value k; then $\mathfrak{A}_k$ must be uncountable for some k. If $k=0$, we have that each $A \in \mathfrak{A}_0$ is itself of second category; but (2.1(2)) each $A \in \mathfrak{B}'$ has the form $Ra+H$ where $a \in E'$ and $H \in \mathcal{K}R$, and if A is of second category then $a \neq o$. So if $\mathfrak{A}_0$ is uncountable, E' would not satisfy the countable chain condition. We may therefore assume that $\mathfrak{A}_k$ is uncountable for some $k \geqq 1$.

By 4.5(7) and 4.8(7) there exist sets $Y_1, Y_2, \cdots \in \mathcal{E}_k$ such that $R - \bigcup Y_n \in \mathfrak{N}_k \subset \mathfrak{N}'$ and $\Phi^{*k}\chi Y_n \leqq 1 \mod \mathfrak{N}'$. For every $A \in \mathfrak{B}'$ we have $[\Phi^{*k}\chi A] = \bigcup_n [\Phi^{*k}\chi(A \cap Y_n)] \mod \mathfrak{N}'$, by 4.9(2) and 4.9(3); hence if $A \in \mathfrak{A}_k$ we have that $[\Phi^{*k}\chi(A \cap Y_n)]$ is of second category for some n. There is therefore some n, which we may assume to be 1, to which uncountably many sets $A \in \mathfrak{A}_k$ correspond; we replace each such A by $A \cap Y_1$, and thus obtain an uncountable family $\mathfrak{A}' \subset \mathfrak{B}'$ of disjoint sets satisfying:

(2) If $A \in \mathfrak{A}'$, then $A \subset Y_1$ and $[\Phi^{*k}\chi A]$ is of second category.

From 4.9(1), there exists for each $A \in \mathfrak{A}'$ a positive integer $n(A)$ and a set $W(A) \in \mathfrak{M}'$ of second category such that $\Phi^{*k}\chi A \geqq (1/n(A))\chi W(A)$. There is a positive integer h such that $n(A) = h$ for uncountably many sets in $\mathfrak{A}'$; we may thus assume further that

(3) If $A \in \mathfrak{A}'$, $\Phi^{*k}\chi A \geqq (1/h)\chi W(A)$.

For each subset $\mathfrak{X} \subset \mathfrak{A}'$, put $W(\mathfrak{X}) = \bigcap\{W(A) | A \in \mathfrak{X}\}$. Then:

(4) If $\mathfrak{X}$ has more than h elements, $W(\mathfrak{X})$ is of first category.

It is enough to prove this when $\mathfrak{X}$ has $h+1$ elements $A_0, A_1, \cdots, A_h$. As these sets are disjoint, we have, modulo $\mathfrak{N}$-negligible functions, from (3), $(h+1)\chi W(\mathfrak{X}) \leqq \sum\{h\Phi^{*k}\chi A_i | 0 \leqq i \leqq h\} = h\Phi^{*k}(\sum \chi A_i)$ (see end of 4.8) $\leqq h\Phi^{*k}\chi(\bigcup A_i) \leqq h\Phi^{*k}\chi Y_1 \leqq h$, proving $W(\mathfrak{X}) \in \mathfrak{N} \subset \mathcal{K}R$.

Consider now the family $\{\mathfrak{Y}\}$ of *maximal* subsets $\mathfrak{Y}$ of $\mathfrak{A}'$ for which $W(\mathfrak{Y})$ is of second category; each $A \in \mathfrak{A}'$ is in at least one $\mathfrak{Y}$ (from (4), since A is itself of second category), and each $\mathfrak{Y}$ contains at most h sets $A \in \mathfrak{A}'$. Further, if $\mathfrak{Y}_1 \neq \mathfrak{Y}_2$, $W(\mathfrak{Y}_1)$ and $W(\mathfrak{Y}_2)$ are in $\mathfrak{M}'$, are both of second category, and have intersection of first category. By an argument similar to that used above for $k=0$, there are only countably many sets $W(\mathfrak{Y})$, and therefore only countably many sets $\mathfrak{Y}$, each with $\leqq h$ elements. Thus $\mathfrak{A}' = \bigcup \mathfrak{Y}$ is countable, giving the desired contradiction.

5.3 *The operator* ϕ. Let $F = F(E)$; we define a mapping ϕ of F^+ in F^+ which will be proved to satisfy the requirements of Theorem 1 (4.1). The elements of F are of the form $\bar{f} = f + \mathcal{Z}(\mathfrak{N})$, where $f \in \mathfrak{F}(\mathfrak{M}')$; here $f + \mathcal{Z}(\mathfrak{N})$ denotes the family of all functions $f+h$ where h is $\mathfrak{N}$-negligible. Since $\mathfrak{B}=\mathfrak{B}'+\mathfrak{N}$, we may require $f\in\mathfrak{F}(\mathfrak{B}')$ here.

Define $\phi\bar{f}=\Phi^*f+\mathrm{Z}(\mathfrak{N})$, where $\bar{f}=f+\mathrm{Z}(\mathfrak{N})$. From 4.8(3)–(5), this definition is single-valued and ϕ maps F^+ in itself. Further, ϕ is countably additive from 4.8(5), and σ-finite from 4.5(7) and 4.8(7); ϕ is thus an F-integral on F (2.6). To verify that ϕ is a cylinder extension of ϕ_0, we first observe that the correspondence $x\leftrightarrow(Rx)+\mathfrak{N}$ is (from 5.1(3)) a finitely additive isomorphism between E' and a finitely additive subalgebra of E which, restricted to E_k, is a complete isomorphism (because $\mathfrak{N}\supset\mathfrak{N}'\supset\mathfrak{N}_k$). We may identify E_k with the complete subalgebra $\{Rx+\mathfrak{N}\mid x\in E_k\}$ of E (equivalently, we take $E_k=(\mathfrak{B}_k+\mathfrak{N})/\mathfrak{N}$); and similarly we may identify F_k with the set of function classes $f+\mathrm{Z}(\mathfrak{N})$, $f\in\mathfrak{F}(\mathfrak{B}_k)$—that is, we realise F_k as $\mathfrak{F}(R,\ \mathfrak{B}_k+\mathfrak{N},\ \mathfrak{N})$. The cylinder mapping of F_k in F is now the identity mapping of F_k. If $f\in\mathfrak{F}(\mathfrak{B}_k)^+$, so that $\bar{f}=f+\mathrm{Z}(\mathfrak{N})$ is a typical element of F_k^+, we again have that $R_k\bar{f}$ is the unique k-continuous function in $\bar{f}$ (compare 4.4)([8]). Now if $f\in\mathfrak{F}(\mathfrak{B}_k)^+$, $\Phi^*f=\Phi^*R_k\bar{f}$ mod $\mathfrak{N}$ by 4.8(4), (5), $=\Phi R_k\bar{f}$ mod $\mathfrak{N}$ by 4.8(7) $=R_k\phi_k f$ mod $\mathfrak{N}$ by definition of Φ, so that $\phi_k\bar{f}=(\Phi^*f)^\sim=\phi f$. That is, ϕ is a cylinder extension of ϕ_k ($k=0,\ 1,\ 2,\ \cdots$), and in particular of ϕ_0.

5.4 *Full homogeneity.* All that remains is to show that ϕ is fully homogeneous. Write $\lambda x=\phi\chi x$ for $x\in E$; thus λ is countably additive and σ-finite. Let $z_0=\bigvee\{z\mid z\in E,\ \lambda z=0\}$, $z_1=e-z_0$, $E^1=\{z\mid z\in E,\ z\leqq z_1\}$. We first show:

(1) Given $x\in E$, there exists $\sigma x\in E^1$ such that, for all $y\in E$,

$$\lambda(y\wedge\sigma x) = (\lambda y)\chi x.$$

(The element σx is in fact unique, but we do not need this.)

For let H be the set of elements $x\in E$ for which such a σx exists. Then

(2) if $x\in H$ and $y\in E$, $\lambda\{y\wedge(z_1-\sigma x)\}=(\lambda y)\chi(-x)$.

For suppose first that $\lambda y\ll\infty$. Then

$$\begin{aligned}\lambda(y) = \lambda(y\wedge z_1) &= \lambda(y\wedge\sigma x)+\lambda(y\wedge(z_1-\sigma x))\\ &= (\lambda y)\chi x+\lambda(y\wedge(z_1-\sigma x)),\end{aligned}$$

so $\lambda(y\wedge(z_1-\sigma x))=(\lambda y)(1-\chi x)=(\lambda y)\chi(-x)$.

In the general case, we know $y=\bigvee y_n$ ($n=1,\ 2,\ \cdots$) where $\lambda y_n\ll\infty$, and the elements y_n can be assumed disjoint. Then $\lambda(y_n\wedge(z_1-\sigma x))=(\lambda y_n)\chi(-x)$, and summation gives (2).

This shows that if $x\in H$ then $-x\in H$, with $\sigma(-x)=z_1-\sigma x$.

Next let $x_n\in H$ ($n=1,\ 2,\ \cdots$), $y\in E$, and suppose $\lambda(y)\ll\infty$. Then $\lambda(y\wedge\bigwedge\sigma x_n)\leqq\lambda(y\wedge\sigma x_n)=(\lambda y)\chi x_n$ for all n, and therefore

(3) $$\lambda(y\wedge\bigwedge\sigma x_n)\leqq(\lambda y)\chi(\bigwedge x_n).$$

([8]) This depends on the observation that if $g\in\mathfrak{F}(\mathfrak{B}_k)$ is $\mathfrak{N}$-negligible, then g is $\mathfrak{N}_k$-negligible. For $g=h$ mod $\mathfrak{N}_k$ where h is k-continuous; being continuous and $\mathfrak{K}R$-negligible, h must be 0.

But $\lambda\{y \wedge (z_1 - \wedge\sigma x_n)\} = \lambda\{y \wedge \vee(z_1 - \sigma x_n)\} \leqq \sum\lambda\{y \wedge (z_1 - \sigma x_n)\} = \sum(\lambda y)\chi(-x_n)$ by (2) $= \lambda y \sum\chi(-x_n)$, and thus $\lambda\{y \wedge (z_1 - \wedge\sigma x_n)\} \leqq \inf\{\lambda y, \lambda y \sum\chi(-x_n)\}$; that is,

$$(4) \qquad \lambda(y \wedge (z_1 - \wedge \sigma x_n)) \leqq (\lambda y)\chi \vee (-x_n).$$

But $\lambda\{y\wedge\wedge\sigma x_n\} + \lambda\{y\wedge(z_1 - \wedge\sigma x_n)\} = \lambda(y\wedge z_1) = \lambda y \ll \infty$. Adding (3) and (4), we see that both inequalities must be equalities; in particular (3) becomes

$$(5) \qquad \lambda(y \wedge\wedge \sigma x_n) = (\lambda y)\chi(\wedge x_n), \qquad \text{if } \lambda y \ll \infty.$$

The restriction $\lambda y \ll \infty$ is easily removed, as before, so (5) shows that $\wedge x_n \in H$, with $\sigma(\wedge x_n) = \wedge\sigma x_n$.

Thus H is a σ-subalgebra of E. Further,

$$(6) \qquad H \text{ contains each } E_k \ (k = 0, 1, 2, \cdots).$$

For (4.2(3)) $c_{k+1,k}\phi_{k+1}$ is a fully homogeneous F_k-integral on F_{k+1}; we apply [4, p. 175, Lemma 4] to its strict form, taking $\sigma x = \pi(z_1, x)$ for $x \in E_{k+1}$, and obtain $c_{k+1,k}\phi_{k+1}\chi_{k+1}(y\wedge\sigma x) = (c_{k+1,k}\phi_{k+1}\chi_{k+1}y)\chi_{k+1}x$, χ_{k+1} denoting the characteristic function in F_{k+1}. We use the same realisations of F_k, F_{k+1} as at the end of 5.3; the cylinder mappings become identities, $\chi_{k+1}y = \chi y$ for $y \in E_{k+1}$, and because ϕ is a cylinder extension of ϕ_{k+1} it follows that $\phi\chi(y\wedge\sigma x) = (\phi\chi y)\chi x$ for $x \in E_{k+1}$; thus $E_k \subset E_{k+1} \subset H$.

Now let $\mathcal{K}$ = family of all sets $X \in \mathcal{B}'$ such that $(X) + \mathfrak{N} \in H$. Then $\mathcal{K}$ is a Borel field containing $\cup\mathcal{E}_k = \mathcal{E}'$, so $\mathcal{K} \supset \mathcal{B}'$. That is, $(X) + \mathfrak{N} \in H$ for all $X \in \mathcal{B}'$, proving $E \subset H$. This establishes (1).

Now, given $x \in E$ and $g \in F^+$ such that $g \leqq \lambda x$, we must find $y \leqq x$ in E such that $\lambda y = g$. We may of course assume $g > 0$; and it is enough to show that there then exists a nonzero $z \leqq x$ in E such that $\lambda z \leqq g$, as then an "exhaustion" argument, based on the countable chain condition (cf. [2, p. 283]), produces the required element y. We may further suppose that $x \leqq a_1$ where $a_1 \in E_1$ and $\lambda a_1 \leqq 1$. For $c_{10}\phi_1 = \phi_1$ is fully homogeneous on F_1 and ϕ extends ϕ_1; hence (as in the proof of 4.6(7)) disjoint elements $a_1, a_2, \cdots \in E_1$ exist such that $\vee a_n = e$ and $\lambda a_n \leqq 1$. Since $\sum\lambda(x\wedge a_n) \geqq g > 0$, there exists n such that $[\lambda(x\wedge a_n)]\wedge[g] \neq o$; we may suppose $n = 1$, and then have $g_1 = \lambda(x\wedge a_1)\wedge g > 0$; we replace x by $x\wedge a_1$ and g by g_1. For some positive integer m we have $g \geqq (1/m)\chi w$ for some nonzero $w \in E$. Because of the full homogeneity of ϕ_1, there exist disjoint elements $b_1, b_2, \cdots, b_m \in E_1$ such that $\vee b_i = a_1$ and $\lambda b_i = (1/m)\lambda a_1$. Since $x \leqq \vee b_i$, $\sum\lambda(b_i\wedge x\wedge\sigma w) = \sum\lambda(b_i\wedge x)\chi w = (\lambda x)\chi w \geqq g\chi w \geqq (1/m)\chi w > 0$. Hence, for some i, $0 < \lambda(b_i\wedge x\wedge\sigma w) \leqq \lambda(b_i\wedge\sigma w) = (\lambda b_i)\chi w \leqq (1/m)\chi w \leqq g$, and we take $z = b_i\wedge x\wedge\sigma w$. This completes the proof of Theorem 1.

6. Extensions of measures on measure algebras.

THEOREM 2. *Let A be a (σ-)subalgebra of a measure algebra (E, μ).*(9) *Let* λ

(9) By saying that (E, μ) is a measure algebra, we imply that μ is σ-finite and positive on E.

be a σ-finite positive measure on A. Then there exists a σ-finite positive measure λ^* on E which extends λ.

In what follows, it is understood that all measures are to be complete and σ-finite, and that the sets and functions used are measurable.

By [3, Theorem 2b, p. 149], (E, μ) has a realisation of the following form. We can realise A algebraically as the measure algebra of a measure space (S, ν) (the measure ν has no simple relation to μ), and can find a measure space (T, m) and a subset K of the product space $S\times T$ (to which we give the usual product measure), in such a way that there is a measure-preserving isomorphism θ of (E, μ) onto a certain Borel field of subsets of K modulo null sets, and further if $a\in A$ then θa is the class of the "cylinder set" $(U\times T)\cap K$, where U is any subset of S in the class a.

By the Radon-Nikodym theorem, there exists a non-negative function f on S such that, for each $U\subset S$, $\lambda(U)=\int_U f(s)d\nu(s)$. (Here $\lambda(U)$ means λa where a is the class of U modulo null sets.) Write $T=\bigcup T_n$ $(n=1, 2, \cdots)$, where the sets T_n are disjoint and $m(T_n)$ is positive and finite. Define

$$P(s) = \int_T \chi_K(s,t)\sum \{(\chi T_n)/2^n m(T_n)\}dm(t);$$

this is defined and $\leqq 1$ for almost all $s\in S$. Further, $P(s)>0$ almost everywhere, since if $P(s)=0$ for all $s\in U$, the set $(U\times T)\cap K$ is null, showing that U is in the class of $o\in A$—that is, $\nu(U)=0$. Now define, for $X\subset S\times T$,

$$\lambda^*(X) = \iint_{S\times T} (f(s)/P(s))(\chi X)\sum (\chi T_n/2^n m(T_n))d\nu(s)dm(t).$$

Then, applying θ, we see that λ^* gives a σ-finite positive measure on E. To show that λ^* extends λ on A, we verify (by a straightforward calculation) that if $U\subset S$, $\lambda^*((U\times T)\cap K)=\lambda U$.

7. **Extensions of operators for measure algebras.** In this section we prove that if we start with a measure algebra in Theorem 1, then we can arrange to end up with a measure algebra. More precisely:

THEOREM 3. *If, in Theorem 1, E_0 is a measure algebra*(9) *with measure m_0, the algebra E can be chosen so that it is also a measure algebra, with measure m extending m_0.*

Proof. Since m_0 is σ-finite on E_0, we can find an equivalent *finite* measure m_0' on E_0; say $m_0'(e)=1$. We use the entire argument of §4 (but *not* of §5), with the following additions. We first observe that E_1 can be taken to be a measure algebra, say with measure m_1. For let $z_0=\bigvee\{x \mid x\in E_0, \phi_0\chi x=0\}$, $z_1=e-z_0$. The "strict form" ϕ_{0s} of ϕ_0 is defined on the function space on the principal ideal $E_0(z_1)=\{x \mid x\in E_0, x\leqq z_1\}$, and its range is the function space on the principal ideal $E_0(\phi 1]$. The construction of E_1 depended in the first instance on applying the result of [4] to ϕ_{0s}; this gives an algebra E_1' containing $E_0(z_1)$ as a subalgebra, and a fully homogeneous *strict* extension ϕ_0^* of ϕ_{0s} to an operator from $F(E_1')$ to $F(E_0[\phi 1])$. We then take E_1 = direct sum of E_1' and $E_0(z_0)$, imbedding E_0 in E_1 in the obvious way; ϕ_1 is defined by $\phi_1 f = c_{01}\phi_{0s}(f\chi z_1)$. By [4, Theorem 5], E_1' is isomorphic to a principal ideal in a product $J \times E_0[\phi 1]$, where J is a numerical measure algebra. By [3; 2; 4], if we give $E_0[\phi 1]$ the measure m_0', then $J\times E_0[\phi 1]$ with the usual product measure induces a (positive, σ-finite) measure (say) m_1 on E_1'. We extend m_1 to E_1 by using m_0' on $E_0(z_0)$. By Theorem 2, there is a (positive) measure m_1' on E_1 which extends m_0'. Note that $m_1' \leqq 1$ on E_1, because $m_1'(e) = m_0'(e) = 1$.

In this way, we may suppose that all the algebras E_k of 4.2 are measure algebras, the measure m_{k+1}' on E_{k+1} extending m_k' on E_k. Their common value gives a *finitely* additive measure m' on E', and hence on the family $\mathcal{E}'$ of sets Rx, $x \in E'$, in the representation space R of E' (cf. 4.3). By the same reasoning as in 4.6, we extend m' to a complete, countably additive measure (still denoted by m') on a Borel field containing $\mathcal{B}'$; note that $m'(R) = 1$. Let $\mathfrak{N}^0$ denote the family of subsets of R with zero m'-measure. We show:

$$\mathfrak{N}^0 \supset \mathfrak{N}_k, \qquad (k = 0, 1, 2, \cdots). \tag{1}$$

For, by 2.3(4), each $X \in \mathfrak{N}_k$ is contained in a set of the form $\xi^{-1}Y$, where $Y \in \mathcal{K}S \cap \mathcal{B}S$. By 3.2, $Y \subset \bigcup_m \bigcap_n Sx_{mn}$ where $x_{mn} \in E_k$, $x_{m1} \geqq x_{m2} \geqq \cdots$ and $\bigwedge_n x_{mn} = o$. Thus $X \subset \xi^{-1}Y \subset \bigcup_m \bigcap_n Rx_{mn}$ by 2.2(1); now, as m_k' is finite, $m_k' x_{mn} \to 0$ as $n \to \infty$, so $m' \bigcap_n Rx_{mn} = 0$ for each m, proving $m'X = 0$ if $X \in \mathfrak{N}_k$.

It follows at once that

$$\mathfrak{N}^0 \supset \mathfrak{N}'. \tag{2}$$

Define $\mathfrak{N} = \{X \mid X \subset Y$ for some $Y \in \mathfrak{M}'$ such that $I(Y) \in \mathfrak{N}^0\}$. It is easily verified that all the properties in 5.1 continue to apply for this modified definition of $\mathfrak{N}$, except that in 5.1(2) we no longer have $\mathfrak{N} \subset \mathcal{K}R$. But instead we have

$$\mathfrak{N} \subset \mathfrak{N}^0, \tag{2'}$$

because if $X \in \mathfrak{N}$ then $X \subset Y \subset I(Y)$ where $I(Y) \in \mathfrak{N}^0$. Hence 5.1(3) continues to hold.

For each $f \in \mathfrak{F}(\mathfrak{M}')^+$, put

$$\psi f = \sum_{1}^{\infty} 2^{-n}\Phi^*(f\chi G_n),$$

the sets $G_1, G_2, \cdots$, being those of 4.6. Then $\psi 1 \leqq 1$, $\psi f \in \mathfrak{F}(\mathfrak{M}')^+$, ψ is linear and countably additive mod $\mathfrak{N}'$, and $[\psi f] = [\Phi^* f]$ mod $\mathfrak{N}'$. Hence $\psi f \in Z(\mathfrak{N}')$ if $f \in Z(\mathfrak{N}')$, and from this an easy induction shows that $[\psi^k f] = [\Phi^{*k} f]$ mod $\mathfrak{N}'$ (where $f \in \mathfrak{F}(\mathfrak{M}')^+$ and $k = 0, 1, 2, \cdots$), and that ψ^k is countably additive mod $\mathfrak{N}'$. Further,

$$[\sum 2^{-k-1}\psi^k\chi X] = I(X) \bmod \mathfrak{N}' \qquad (X \in \mathfrak{M}'), \tag{3}$$

because (modulo $\mathfrak{N}'$) $I(X) = [\sum\Phi^{*k}\chi X] = \bigcup[\Phi^{*k}\chi X] = \bigcup[\psi^k\chi X]$ $=\bigcup[2^{-k-1}\psi^k\chi X] = [\sum 2^{-k-1}\psi^k\chi X]$.

Now define, for $X \in \mathfrak{M}'$,

$$m^*X = \int_R \sum 2^{-k-1}\psi^k\chi X dm'.$$

The integrand is $\mathfrak{M}'$-measurable, non-negative, and $\leqq 1$, so m^* is well defined and is a finite, countably additive measure on $\mathfrak{M}'$. We complete this measure as usual, still calling it m^*, and show

$$m^*Y = 0 \text{ if and only if } Y \in \mathfrak{N}. \tag{4}$$

For if $m^*Y=0$, we have $Y \subset X$ where $X \in \mathfrak{M}'$ and $m^*X=0$. In the definition of m^*X, the integrand must be zero almost everywhere (m'), which from (3) and (2) gives $I(X) \in \mathfrak{N}^0$ and therefore $Y \in \mathfrak{N}$. Conversely, if $Y \in \mathfrak{N}$, $Y \subset X \in \mathfrak{M}'$ where $I(X) \in \mathfrak{N}^0$, and so $m^*X=0$.

The algebra E is now defined exactly as in 5.2, except that we use the new meaning of $\mathfrak{N}$; and the measure m^* on $\mathfrak{M}'+\mathfrak{N}$ induces a positive σ-finite measure m^* on E. The proof of 5.2(1) no longer applies (as there we used $\mathfrak{N} \subset \mathfrak{K}R$), but the result itself (the countable chain condition) is a trivial consequence of the existence of m^*. The operator ϕ is defined just as in 5.3; the arguments in 5.3 and 5.4 apply unchanged, so that ϕ is a fully homogeneous cylinder extension of ϕ_0. Finally, by Theorem 2, we replace the measure m^* on E by a (positive, σ-finite) measure m which extends m_0 on E_0, and the proof is complete.

References

1. E. Hopf, *Ergodentheorie*, Berlin, Springer, 1937.

2. D. Maharam, *The representation of abstract measure functions*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 279–330.

3. ———, *Decompositions of measure algebras and spaces*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 142–160.

4. ———, *The representation of abstract integrals*, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 154–184.

5. ———, *On kernel representation of linear operators*, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 229–255.

6. M. H. Stone, *Boundedness properties in function lattices*, Canad. J. Math. vol. 1 (1949) pp. 176–186.

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